Name: _____ PID: _____

Math 140C: Final Exam
Foundations of Real Analysis

- You have 3 hours. No books and notes are allowed.
- You may quote any result stated in the textbook or in class.
- State carefully the hypothesis and conclusion of any result that you use.
- You may not use homework problems (without proof) in your solutions.

1. (10 points)

(a) (3 points) Define what it means for a function $f : \mathbb{R}^2 \to \mathbb{R}$ to be differentiable at $(0,0)$.

(b) (7 points) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that the partial derivatives $\frac{\partial f}{\partial x}(x,y)$ and $\frac{\partial f}{\partial y}(x,y)$ exist and satisfy $|\frac{\partial f}{\partial x}(x,y)| \leqslant |x|$ and $|\frac{\partial f}{\partial y}(x,y)| \leqslant |y|$, for all $(x,y) \in \mathbb{R}^2$.
Prove that f is differentiable at $(0,0)$ and $f'(0,0) = 0$.

(a) Adapt Definition 9.11 in Rudin with $\vec{x} = (0,0)$.

If there exists a linear transformation A of $\mathbb{R}^2$ into $\mathbb{R}$ such that

$$\lim_{\vec{h} \to 0} \frac{|f(\vec{0}+\vec{h}) - f(\vec{0}) - A\vec{h}|}{|\vec{h}|} = 0$$

then we say that f is differentiable at $(0,0)$.

(b) It suffices to show that

$$\lim_{(h_1,h_2)\to(0,0)} \frac{|f(h_1,h_2) - f(0,0) - (0,0)\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}|}{\sqrt{h_1^2 + h_2^2}} = 0.$$

Using the mean value theorem,

$$|f(h_1,h_2) - f(0,0)| \leq |f(h_1,h_2) - f(h_1,0)| + |f(h_1,0) - f(0,0)|$$

$$= \left|\frac{\partial f}{\partial y}(h_1,\xi)\cdot h_2\right| + \left|\frac{\partial f}{\partial y}(\eta,0)\cdot h_1\right| \quad \text{for some } \xi \in [0,h_2] \text{ or } [h_2,0] \text{ and } \eta \in [0,h_1] \text{ or } [h_1,0]$$

$$\leqslant |\xi \cdot h_2| + |\eta \cdot h_1| \leqslant h_2^2 + h_1^2$$

Thus $\lim_{(h_1,h_2)} \frac{|f(h_1,h_2) - f(0,0)|}{\sqrt{h_1^2+h_2^2}} = 0$

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that the partial derivatives $\frac{\partial f}{\partial x}(x,y)$, $\frac{\partial f}{\partial y}(x,y)$ and the second-order partial derivative $\frac{\partial^2 f}{\partial x \partial y}(x,y)$ exist at every $(x,y) \in \mathbb{R}^2$.

(a) (5 points) Assume that $\frac{\partial f}{\partial x \partial y}$ is continuous at $(0,0)$. Prove that

$$\lim_{t \to 0} \frac{f(t,t^2) - f(t,0) - f(0,t^2) + f(0,0)}{t^3} = \frac{\partial^2 f}{\partial x \partial y}(0,0).$$

(b) (5 points) Assume that $\frac{\partial^2 f}{\partial x \partial y}(x,y) = 0$, for every $(x,y) \in \mathbb{R}^2$. Prove that there exist two differentiable functions $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ such that $f(x,y) = g(x) + h(y)$, for every $(x,y) \in \mathbb{R}^2$.

(a) (You cannot exchange two limits) Use mean value theorem,

$$\lim_{t\to 0} \frac{f(t,t^2) - f(t,0) - f(0,t^2) + f(0,0)}{t} = \lim_{t\to 0} \frac{t^2 \frac{\partial f}{\partial y}(t,\eta) - t^2 \frac{\partial f}{\partial y}(0,\eta)}{t^3}$$

$\eta \in [0, t^2]$

$$= \lim_{t\to 0} \frac{\frac{\partial f}{\partial y}(t,\eta) - \frac{\partial f}{\partial y}(0,\eta) + \frac{\partial f}{\partial y}(0,\eta) - \frac{\partial f}{\partial y}(0,0)}{t}$$

$\eta - 0$ has order t^2.

$$= \lim_{t\to 0} \left(\frac{\partial^2 f}{\partial x \partial y}(t',\eta) + \frac{1}{t} \frac{\partial^2 f}{\partial x \partial y}(0,t'')(\eta - 0) \right) \quad t' \in [0,t]$$

or $(0-\eta)$ $\quad t'' \in [0,\eta]$ or $[\eta, 0]$

$$= \frac{\partial^2 f}{\partial x \partial y}(0,0) + 0 = \frac{\partial^2 f}{\partial x \partial y}(0,0) \text{ by continuity of } \frac{\partial^2 f}{\partial x \partial y} \text{ at } (0,0).$$

(b) Since $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(x,y) = 0$ for all (x,y), we have

$$\frac{\partial f}{\partial y}(x,y) = \tilde{h}(y).$$

Then $f(x,y) = f(x,0) + \int_0^y \frac{\partial f}{\partial t}(x,t)\, dt$

$$= f(x,0) + \int_0^y \tilde{h}(t)\, dt$$

$g(x) = f(x,0)$

$h(y) = \int_0^y \tilde{h}(t)\, dt.$

So we can write $f(x,y) = g(x) + h(y)$

4. (10 points) Consider the measurable space $X = \mathbb{R}$ in which the σ-ring $\mathscr{M}$ is the set of all Lebesgue measurable subsets of $\mathbb{R}$.

(a) (3 points) Assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. Prove that f is measurable.

(b) (3 points) Assume that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function. Prove that the derivative $f' : \mathbb{R} \to \mathbb{R}$ is measurable.

(c) (4 points) Assume that $f : \mathbb{R} \to \mathbb{R}$ is a monotonically increasing function. Prove that f is measurable.

(a) Since f is continuous, $f^{-1}(-\infty, c)$ is an open set since $(-\infty, c)$ is open. And open sets are measurable. Thus f is measurable.

(b) Define $g_n(x) = \dfrac{f(x+\frac{1}{n}) - f(x)}{\frac{1}{n}}$ for $n = 1, 2, 3, \cdots$.

It's easy to see $g_n(x)$ is measurable for every n.

$f'(x) = \lim_{n\to\infty} g_n(x)$ implies $f'(x)$ is measurable.

(c) Observe that if f is increasing, then $f^{-1}(-\infty, c)$ is either empty or an interval.

Thus f is measurable.

5. (10 points) Assume the measure in this problem is the Lebesgue measure.

(a) (5 points) Let $A = \cup_{n=1}^{\infty}[n, n+\frac{1}{n^2}]$.
Prove that if a function $f : A \to \mathbb{R}$ satisfies $f \in \mathscr{L}^2$ on A then $f \in \mathscr{L}$ on A.

(b) (5 points) Let $B = \cup_{n=1}^{\infty}[n, n+\frac{1}{\sqrt{n}}]$.
Give an example of a function $f : B \to \mathbb{R}$ such that $f \in \mathscr{L}^2$ on B and $f \notin \mathscr{L}$ on B.

(a) Since $\int_A 1^2 d\mu = \mu(A) = \mu\left(\bigcup_{n=1}^{\infty}[n, n+\frac{1}{n^2}]\right) = \sum_{n=1}^{\infty}\mu[n, n+\frac{1}{n^2}]$

$= \sum_{n=1}^{\infty}\frac{1}{n^2} < +\infty$, $\quad 1 \in \mathcal{L}(\mu)$.

By the Schwarz inequality

$$\int_A f\,d\mu \le \int_A |f|\,d\mu \le \left(\int_A f^2\,d\mu\right)^{\frac{1}{2}} \cdot \left(\int_A 1^2 d\mu\right)^{\frac{1}{2}} < +\infty$$

Thus $f \in \mathcal{L}(\mu)$.

(b) Let $f = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}} K_{[n, n+\frac{1}{\sqrt{n}}]}$, then $f^2 = \sum_{n=1}^{\infty}\frac{1}{n} K_{[n, n+\frac{1}{\sqrt{n}}]}$

then $\int_B f^2\,d\mu = \int_B \sum_{n=1}^{\infty}\frac{1}{n} K_{[n, n+\frac{1}{\sqrt{n}}]}\,d\mu = \sum_{n=1}^{\infty}\frac{1}{n}\int_B K_{[n, n+\frac{1}{\sqrt{n}}]}\,d\mu$

$= \sum_{n=1}^{\infty}\frac{1}{n}\mu[n, n+\frac{1}{\sqrt{n}}] = \sum_{n=1}^{\infty}\frac{1}{n}\cdot\frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}}} < +\infty$.

$$\int_B f\,d\mu = \int_B \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}} K_{[n, n+\frac{1}{n}]}\,d\mu = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\int_B K_{[n, n+\frac{1}{\sqrt{n}}]}\,d\mu$$

$$= \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\cdot\frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty}\frac{1}{n} = +\infty$$

Hence $f \in \mathcal{L}^2(\mu)$ but $f \notin \mathcal{L}(\mu)$

6. (10 points) Let $f : [a, b] \to \mathbb{R}$ be a function such that $f \in \mathscr{L}$ on $[a, b]$.
Define $F : [a, b] \to \mathbb{R}$ by letting $F(x) = \int_a^x f(t)\, dt$.

(a) (5 points) Prove that F is continuous on $[a, b]$.

(b) (5 points) Assume that f is Riemann integrable on $[a, b]$. Prove that $F'(x)$ exists and $F'(x) = f(x)$, almost everywhere on $[a, b]$.

(a) For any $x \in [a,b]$, Let x_n be any sequence that converges to x.

Define $f_n(x) = K_{[a,x_n]} f(x)$. Then $f_n(x)$ converges to $f(x) \cdot K_{[a,x]}$ except possibly at the point x, which is of measure 0. Also $|f_n(x)| \le |f(x)|$, thus we can apply the Lebesgue's dominated convergence theorem to get $\int f(t) \cdot K_{[a,x]}\, dt = \lim_{n\to\infty} \int K_{[a,x_n]} f(t)\, dt$

i.e. $F(x) = \int_a^x f(t)\, dt = \lim_{n\to\infty} \int_a^{x_n} f(t)\, dt = \lim_{n\to\infty} F(x_n)$

Thus $F(x)$ is continuous on $[a, b]$.

(b) We know by theorem 6.20 that $F'(x) = f(x)$ at every point where $f(x)$ is continuous. And theorem 11.33 shows that if f is Riemann integrable on $[a,b]$, then it is continuous almost everywhere on $[a, b]$. Hence the result follows.

7. (10 points) Let $f : [0,2] \to \mathbb{R}$ be a function such that $f \in \mathscr{L}^2$ on $[0,2]$. Let $g : [0,1] \to \mathbb{R}$ be a function such that $g \in \mathscr{L}^2$ on $[0,1]$. Let $A \subset [0,1]$ be a Lebesgue measurable set such that $m(A) > 0$, where m denotes the Lebesgue measure. For $t \in \mathbb{R}$, let $A+t = \{x+t | x \in A\}$.

(a) (4 points) Prove that $\lim_{n\to\infty} \int_0^1 |f(x+\frac{1}{n}) - f(x)|^2 \, dx = 0$.

(b) (3 points) Prove that $\lim_{n\to\infty} \int_0^1 f(x+\frac{1}{n}) g(x) \, dx = \int_0^1 f(x)g(x) \, dx$.

(c) (3 points) Prove that there exists $n \geqslant 1$ such that $(A+\frac{1}{n}) \cap A \neq \emptyset$.

(a) If $f(x)$ is continuous on $[0,2]$, then $\lim_{n\to\infty} |f(x+\frac{1}{n}) - f(x)| = 0$

and $|f(x+\frac{1}{n}) - f(x)| \leq 2 \cdot \sup_{[0,2]} f \in \mathcal{L}[0,2]$ since $m([0,2]) < +\infty$.

Lebesgue's dominated thm then implies $\lim \int_0^1 |f(x+\frac{1}{n}) - f(x)|^2 dx = 0$.

If $f(x)$ is not continuous, we use approximation by continuous functions.

$\forall \varepsilon > 0$, $\exists\ h(x) \in C([0,2])$ s.t. $\int_0^2 |f(x) - h(x)|^2 < \varepsilon$.

This implies $\int_0^1 |f(x+\frac{1}{n}) - h(x+\frac{1}{n})|^2 dx < \varepsilon$. Therefore

$$\left(\int_0^1 |f(x+\tfrac{1}{n}) - f(x)|^2 dx \right)^{\frac{1}{2}} = \left(\int_0^1 |f(x+\tfrac{1}{n}) - h(x+\tfrac{1}{n}) + h(x+\tfrac{1}{n}) - h(x) + h(x) - f(x)|^2 dx \right)^{\frac{1}{2}}$$

$$\leq \left(\int_0^1 |f(x+\tfrac{1}{n}) - h(x+\tfrac{1}{n})|^2 dx \right)^{\frac{1}{2}} + \left(\int_0^1 |h(x+\tfrac{1}{n}) - h(x)|^2 dx \right)^{\frac{1}{2}} + \left(\int_0^1 |h(x) - f(x)|^2 dx \right)^{\frac{1}{2}}$$

$$\leq \varepsilon^{\frac{1}{2}} + \left(\int_0^1 |h(x+\tfrac{1}{n}) - h(x)|^2 dx \right)^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}}$$

The second term approaches 0 as $n \to \infty$ because h is continuous.

Thus $\lim_{n\to\infty} \int_0^1 |f(x+\frac{1}{n}) - f(x)|^2 dx = 0$.

(b) $\int_0^1 f(x+\frac{1}{n})\, g(x)\, dx - \int_0^1 f(x)\, g(x)\, dx$

$\leq \int_0^1 |f(x+\frac{1}{n}) - f(x)|\, |g(x)|\, dx$

$\leq \left(\int_0^1 |f(x+\frac{1}{n}) - f(x)|^2 dx\right)^{\frac{1}{2}} \left(\int_0^1 |g(x)|^2 dx\right)^{\frac{1}{2}}$

Since $g \in L^2[0,1]$, the above tends to 0 as $n \to \infty$.

(c) Applying part (b) to $f = K_A$, $g = K_A$, $f(x+\frac{1}{n}) = K_{A+\frac{1}{n}}$

and we obtain

$$\lim_{n\to\infty} \int_0^1 K_A \cdot K_{A+\frac{1}{n}}\, dx = \int_0^1 K_A^2\, dx = \int_0^1 K_A\, dx = m(A) > 0$$

If $A \cap (A+\frac{1}{n}) = \phi$ for every n, then

$$\lim_{n\to\infty} \int_0^1 K_A \cdot K_{A+\frac{1}{n}}\, dx = \lim_{n\to\infty} \int_{A\cap(A+\frac{1}{n})} dx = \lim_{n\to\infty} m\left(A \cap (A+\tfrac{1}{n})\right) = 0$$

which is a contradiction.

Thus $\exists\, n$ s.t. $A \cap (A+\frac{1}{n}) \neq \phi$.

8. (10 points) Let X be a measurable space, in which $\mathscr{M}$ is the σ-ring of measurable sets and μ is the measure. Let $f : X \to \mathbb{R}$ be a function such that $f \in \mathscr{L}(\mu)$. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of sets such that $A_n \in \mathscr{M}$, for all $n \geqslant 1$.

(a) (2 points) Prove that if $A \in \mathscr{M}$, then $\int_A f \, d\mu = \int_X f \, 1_A \, d\mu$. ✓

(b) (3 points) Assume that $\sum_{n=1}^{\infty} \mu(A_n) < +\infty$. Prove that $\lim_{n\to\infty} 1_{A_n}(x) = 0$, almost everywhere on X.

(c) (3 points) Assume that $\sum_{n=1}^{\infty} \mu(A_n) < +\infty$. Prove that $\lim_{n\to\infty} \int_{A_n} f \, d\mu = 0$.

(d) (2 points) Assume that $\lim_{n\to\infty} \mu(A_n) = 0$. Prove that $\lim_{n\to\infty} \int_{A_n} f \, d\mu = 0$.

(a) $\int_X f \cdot K_A \, d\mu = \int_A f \cdot K_A \, d\mu + \int_{A^c} f \cdot \underbrace{K_A}_{=0} \, d\mu = \int_A f \cdot K_A \, d\mu$

(b) By Fatou's Lemma, $\int_X \lim K_{A_n}(x) \, d\mu \leqslant \liminf \int_X K_{A_n}(x) \, d\mu$

$= \liminf \mu(A_n)$.

Since $\sum_{n=1}^{\infty} \mu(A_n) < +\infty$, we have $\lim \mu(A_n) = 0$.

Thus $\lim K_{A_n}(x) = 0$ a.e.

(c) Note that $K_{A_n}(x) \cdot f(x) \to 0$ a.e and $|K_{A_n}(x) \cdot f(x)| \leqslant |f| \in L(\mu)$

Lebesgue's dominated convergence theorem implies

$$\lim_{n\to\infty} \int_X K_{A_n(x)} f(x) \, d\mu = \lim_{n\to\infty} \int_{A_n} f \, d\mu = \int_X \lim K_{A_n}(x) f(x) \, d\mu = 0.$$

(d). Noting that in (b), $\sum_{n=1}^{\infty} \mu(A_n) < +\infty$ can be weakened to $\lim_{n\to\infty} \mu(A_n) = 0$.

Combining the proofs of (b) and (c) and then the results follows.

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1		out of 10 points
2		out of 10 points
3		out of 10 points
4		out of 10 points
5		out of 10 points
6		out of 10 points
7		out of 10 points
8		out of 10 points
Total		out of 40 points